

Lagrangian dynamics of thermal tracer particles in Navier-Stokes fluids

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A key issue in fluid dynamics is the definition of the phase-space Lagrangian dynamics characterizing prescribed ideal fluids (i.e., continua), which is related to the dynamics of so-called *ideal tracer particles* moving in the same fluids. These are by definition particles of infinitesimal size which do not produce significant perturbations of the fluid fields and do not interact among themselves. In this work we point out that the phase-space Lagrangian description of incompressible Navier-Stokes fluids can be achieved by means of a particular subset of ideal tracer particles, denoted as thermal particles. For these particles the magnitude of their relative velocities - with respect to the local fluid velocity - is solely determined by the kinetic pressure, in turn, uniquely related to the fluid pressure. The dynamics of thermal tracer particles is shown to generate the time-evolution of the fluid fields by means of a suitable statistical model. The result is reached introducing a 1-D statistical description of the fluid exclusively based on the ensemble of thermal tracer particles. In particular, it is proven that the statistic of thermal particles can be uniquely defined requiring that for these particles the directions of their initial relative velocities are defined by a suitable family of random coplanar unit vectors.

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1 - INTRODUCTION

The phase-space Lagrangian description of incompressible Navier-Stokes (NS) fluids is an essential ingredient of turbulence theory [1, 2]. Turbulence, in fact, can be viewed as the manifestation of stochastic behavior of the underlying phase-space dynamical system which advances in time the state of the fluid (the so-called *NS dynamical system* [3]). It is therefore important to provide an insight into the phase-space dynamics which characterizes these systems [4, 5, 6].

In the framework of the theory of continua, for a given fluid system, there is a unique minimal class of such dynamical systems which provide - via the introduction of an appropriate phase-space probability density function (PDF) - the time evolution of the complete set of fluid fields describing the same fluid. In turbulence theory this phase space is usually identified with the set $\Gamma = \Omega \times U$, with $\dim(\Gamma) = 6$, denoted as *restricted phase-space* (with $U \equiv \mathbb{R}^3$ denoting the velocity space and $\bar{\Gamma} = \bar{\Omega} \times U$ the closure of Γ). Therefore it is natural to seek possible phase-space representations of this type for fluid systems.

Although phase-space descriptions of incompressible NS fluids have been around for a long time, starting from the historical work of Hopf (Hopf, 1952 [7]), Edwards (Edwards, 1964 [8]) and Rosen (Rosen, 1971 [9]), a fundamental issue concerns the construction of phase-space approaches exclusively based on *classical statistical mechanics* (CSM). This concerns, in particular, the search of so-called *inverse kinetic theories* (IKT) able to yield

the complete set of fluid fields describing the fluid. In these approaches, denoted as *complete IKT's* [10], the fluid is described in terms of a statistical ensemble of point-particles, called *ideal tracer particles*, whose phase-space trajectories are defined by the general solution of the initial-value problem of the form

$$\begin{cases} \frac{d}{dt}\mathbf{r}(t) = \mathbf{v}(t), \\ \frac{d}{dt}\mathbf{v}(t) = \mathbf{F}(\mathbf{r}(t), t; f), \\ \mathbf{r}(t_o) = \mathbf{r}_o, \\ \mathbf{v}(t_o) = \mathbf{v}_o, \end{cases} \quad (1)$$

where \mathbf{F} is a suitable vector field (*mean-field force*) and the state-vector $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ spans by assumption the phase space $\Gamma \equiv \Omega \times U$. For them, as pointed out elsewhere [5, 6] and unlike traditional approaches [11, 12, 13, 14, 15], the dynamics (defined in terms of \mathbf{F}) can be in principle uniquely established based exclusively on first principles. As a result, by construction, in a complete IKT, ideal tracer particles:

- uniquely determine, by means of the 1-point statistics, the time evolution of the fluid fields;
- interact with the fluid via a suitable mean-field force \mathbf{F} ;
- do not perturb the fluid fields;
- are collisionless (i.e., they do not interact among themselves);

- are subject to a mean-field force defined in such a way that the dynamics of the ensemble of tracer particles reproduces exactly, via a suitable statistics, the dynamics of the fluid fields. In particular, the complete set of fluid fields is determined in terms of the 1-point PDF.

An IKT of this type, in which tracer particles can have *arbitrary initial velocities* (i.e., which belong to the whole velocity space $U \equiv \mathbb{R}^3$), has been recently established for incompressible thermofluids (see in particular, Tassarotto et al., [6]). In particular, in the case of an incompressible and isothermal NS fluid it is immediate to show that the dynamics of tracer particles generally differs from that of the corresponding fluid elements [5, 6, 16]. The difference is intrinsic, i.e., it is produced by the mean-field force acting on the tracer particles (which differs from the local force density acting on the fluid elements), so that generally the fluid velocity is *not* a solution of the NS dynamical system.

A peculiar characteristic of a complete IKT is, however, that the mean-field force \mathbf{F} contains an arbitrary scalar parameter (a) [17]. This feature, which - nevertheless - is not reflected by the fluid fields (which remain by construction independent of the same parameter), can also be interpreted as the non-uniqueness in the definition of the NS dynamical system.

An interesting question is, however, whether a proper subset of solutions (of the NS dynamical system) exists which can be used to characterize uniquely incompressible NS fluids and resolve also uniquely the previous indeterminacy issue. This refers, in particular, to the possibility of identifying a proper subset of the family of tracer particles, characterized by a phase-space of lower dimension (i.e., smaller than 6) and whose states uniquely determine *exactly* - via a suitable statistics - both the complete set of fluid fields and their time evolution.

Goals of the investigation

Purpose of this paper is to point out the discovery of a family of tracer particles denoted as *thermal tracer particles*, which have all such features. These tracer particles are characterized by relative velocities belonging to a subset of the velocity space U of dimension 1. As a main consequence, it is shown (see THM.1) that the dynamics of these particles uniquely determines - via a suitable statistics - the time-evolution of the NS fluid. The ensemble of these particles is defined by the requirement that for each tracer particle the magnitude of its relative velocity (with respect to the local fluid velocity) is proportional to the fluid pressure. More precisely, denoting by $\mathbf{x}(t) = \{\mathbf{r}(t), \mathbf{v}(t)\}$ the phase-space trajectory of a generic thermal tracer particle, with $\mathbf{r} = \mathbf{r}(t)$ and

$\mathbf{v} = \mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t)$ its position and velocity, $\mathbf{V}(\mathbf{r}, t)$ the local fluid velocity and $\mathbf{u} = \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$ the relative particle velocity, we intend to prove that $\mathbf{x}_{th}(t) = \{\mathbf{r}(t), \mathbf{v}(t)\}$, with

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{v}(t), \quad (2)$$

$$\mathbf{v}(t) = \mathbf{V}(\mathbf{r}, t) + \mathbf{u}(t), \quad (3)$$

where

$$\mathbf{u}(t) = \mathbf{n}(\mathbf{r}, t)u(t), \quad (4)$$

$$u(t) = v_{th}(\mathbf{r}, t) \equiv \sqrt{\frac{2p_1(\mathbf{r}, t)}{\rho_o}}, \quad (5)$$

is a particular solution of the NS dynamical system. In particular, v_{th} denotes the *thermal velocity* produced by the kinetic pressure $p_1(\mathbf{r}, t)$ [which is, in turn, related to the fluid pressure, see Eq.(24) below], while $\mathbf{n}(\mathbf{r}, t)$ is the rotating unit vector which satisfies the initial-value problem

$$\begin{cases} \frac{d}{dt}\mathbf{n}(\mathbf{r}, t) = \mathbf{n}(\mathbf{r}, t) \times \boldsymbol{\Omega}(\mathbf{r}, t), \\ \mathbf{n}(\mathbf{r}(t_o), t_o) = \mathbf{n}(\mathbf{r}_o, t_o). \end{cases} \quad (6)$$

Here $\boldsymbol{\Omega}(\mathbf{r}, t)$ is the angular velocity

$$\boldsymbol{\Omega}(\mathbf{r}, t) = a\xi, \quad (7)$$

produced only by the *fluid vorticity* $\xi = \nabla \times \mathbf{V}$, while the unit vector $\mathbf{n}(\mathbf{r}, t)$ is subject to the requirement

$$\mathbf{n}(\mathbf{r}, t) \cdot \nabla p_1(\mathbf{r}, t) = 0 \quad (8)$$

(*constraint of orthogonality*). In Eq.(7) $a \equiv a(\mathbf{r}, t)$ is the real scalar parameter which characterizes the mean-field force \mathbf{F} [17]. The arbitrariness of a implies that the definition of the angular velocity $\boldsymbol{\Omega}(\mathbf{r}, t)$ is non-unique, a feature which can be viewed as a kind of *kinematic indeterminacy* of the theory. In Ref.[17] a was determined based on phenomenological arguments as

$$a = 1/2. \quad (9)$$

Here we intend to show, however, that the requirement of the existence of the solutions of the type (2)-(8) actually imposes also a *kinematic constraint* on the same parameter, i.e., that there results also

$$a\mathbf{n} \cdot \boldsymbol{\omega} + \mathbf{n} \cdot \nabla \frac{d}{dt}p_1 = 0, \quad (10)$$

where $\boldsymbol{\omega} \equiv (\nabla \times \mathbf{V}) \times \nabla p_1$. In other words (see THM.1) the parameter a can also be defined in such a way that, if at the initial time t_o , $\mathbf{n}(\mathbf{r}_o, t_o)$ satisfies the constraint of orthogonality (with respect to $\nabla_o p_1(\mathbf{r}_o, t_o)$), then at any time t , $\mathbf{n}(\mathbf{r}(t), t)$ will satisfy the same constraint provided (10) is satisfied too. Therefore, in such a case the relative velocity of a generic thermal tracer particle $\mathbf{u}(t)$ is

a rotating vector orthogonal to $\nabla p_1(\mathbf{r}, t)$. In particular, it is found that while its magnitude $u(t)$ depends, according to Eq.(5), only on the kinetic pressure $p_1(\mathbf{r}, t)$, the angular rotation velocity $\boldsymbol{\Omega}(\mathbf{r}, t)$, which characterizes the direction of the relative velocity, is produced by fluid vorticity.

Main consequence of the theory here presented is that dynamics of an incompressible NS fluid, i.e., the fluid fields and the complete set of fluid equations, can be represented solely in terms of the kinetic state of the thermal tracer particles (see THM.2). The goal is reached by proving that the dynamics of these particles uniquely generates, via a suitable 1-D statistics, the time-evolution of the fluid fields.

Scheme of presentation

The general framework (of this work) is provided by the phase-space representation for an incompressible NS fluid previously developed (see in particular Tessarotto and coworkers [3, 5, 16, 18]). In Sec. 2 the relevant aspect of the theory are recalled, with particular reference to the so-called *Hopf-Rosen-Edwards* (HRE) approach [7, 8, 9] (see also Monin and Yaglom, 1975 [19] and Pope, 2000 [1]) and the IKT approach developed in Refs.[3, 18]. This permits us to identify the NS dynamical system, via the construction of a suitable mean-field force. After displaying the corresponding phase-space Lagrangian formulation (in Sec. 3), the existence of the thermal tracer-particles solutions is proven in THM.1 (see Section 4). Finally the statistics of thermal tracer particles is investigated in Sec. 5 (THM.2). This is shown to depend only on the direction of the tracer-particle relative velocity, defined by a coplanar family of unit vectors.

2 - PHASE-SPACE APPROACHES TO FLUID DYNAMICS

A fundamental aspect of fluid dynamics is the construction of phase-space approaches based on *classical statistical mechanics* (CSM). Indeed, phase-space techniques are well known both in classical and quantum fluid dynamics. In this connection, a particular viewpoint is represented by the class of so-called *inverse problems*, involving the search of an inverse kinetic theory (IKT) able to yield the complete set of fluid equations for the fluid fields, via the introduction of a suitable phase-space PDF $f(\mathbf{x}, t; Z)$ on the restricted phase-space Γ [18]. This requires, by assumption, that the PDF must depend at least functionally, i.e., via appropriate "moments", on the (deterministic) fluid fields $\{Z\}$ characterizing the fluid (or more generally on a suitable subset of $\{Z\}$).

A particular realization for $f(\mathbf{x}, t; Z)$, holding for an incompressible NS fluid, is provided by the well-known

Hopf-Rosen-Edwards (HRE) approach [7, 8, 9] (see also Monin and Yaglom, 1975 [19] and Pope, 2000 [1]). In this case $f(\mathbf{x}, t; Z)$ satisfies the Liouville equation [or *inverse kinetic equation*]

$$L(\mathbf{r}, \mathbf{v}, t; Z, f)f(\mathbf{x}, t; Z) = 0, \quad (11)$$

with $f(\mathbf{x}, t; Z) \equiv f_H(\mathbf{x}, t; Z)$ denoting the PDF

$$f_H(\mathbf{x}, t; Z) = \delta(\mathbf{v} - \mathbf{V}(\mathbf{r}, t)) \quad (12)$$

and $L(\mathbf{r}, \mathbf{v}, t; Z, f)$ a suitable Liouville streaming operator. Here the notation is standard [3]. Thus $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ is a state-vector which spans the phase space $\Gamma \equiv \Omega \times U$, Ω and $U \equiv \mathbb{R}^3$ denoting respectively the configuration and velocity spaces. Moreover, $L(\mathbf{r}, \mathbf{v}, t; Z, f)$ reads

$$\begin{aligned} L(\mathbf{r}, \mathbf{v}, t; Z, f) &\equiv \frac{\partial}{\partial t} \cdot + \frac{\partial}{\partial \mathbf{x}} \cdot \{ \mathbf{X}(\mathbf{x}, t; Z) \cdot \} \equiv \quad (13) \\ &\equiv \frac{\partial}{\partial t} \cdot + \mathbf{v} \cdot \nabla \cdot + \frac{\partial}{\partial \mathbf{v}} \cdot \{ \mathbf{F}(\mathbf{x}, t; Z) \cdot \} \end{aligned}$$

and $\mathbf{X}(\mathbf{x}, t; Z)$ is the vector field

$$\mathbf{X}(\mathbf{x}, t; Z) = \{ \mathbf{v}, \mathbf{F} \}. \quad (14)$$

In particular, in the HRE approach \mathbf{F} is identified with $\mathbf{F} \equiv \mathbf{F}_H$, i.e., the total fluid force per unit mass acting on each fluid element [see Eq.(79) in Appendix A]. It follows by construction that:

- the first two velocity moments of $f(\mathbf{x}, t; Z)$ are necessarily

$$\int_U d\mathbf{v} f(\mathbf{x}, t; Z) = 1, \quad (15)$$

$$\int_U d\mathbf{v} \mathbf{v} f(\mathbf{x}, t; Z) = \mathbf{V}(\mathbf{r}, t); \quad (16)$$

- the corresponding moment equations obtained above coincide manifestly with the complete set of fluid equations [see Eqs.(73) and (74)];
- since the PDF $f(\mathbf{x}, t; Z)$ of Eq.(12), is independent of the fluid pressure, $p(\mathbf{r}, t)$ cannot, manifestly, be represented as a moment of the same PDF.

However, a more general viewpoint is represented by the class of so-called *complete IKT's* able to yield as moments of the PDF the *whole set of fluid fields* $\{Z\}$ which appear in the fluid equations. A theory of this type has been recently developed by Tessarotto and coworkers [3, 18]. In the case of INSE the theory must include necessarily, among the PDF moments, also the fluid pressure. This requires (*Principle of correspondence*) that, besides Eqs.(15) and (16), there must exist a suitable phase-space function $G(\mathbf{x}, t)$ for which it results identically in $\bar{\Omega} \times I$

$$p(\mathbf{r}, t) = \int_U d\mathbf{v} f(\mathbf{x}, t; Z) G(\mathbf{x}, t). \quad (17)$$

In particular, one can show that a possible choice is provided by the position [3]

$$G(\mathbf{x}, t) = \frac{\rho_o u^2}{3} - p_0(t) - \phi(\mathbf{r}, t), \quad (18)$$

where \mathbf{u} is the relative velocity $\mathbf{u} = \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$ and $p_0(t) > 0$ is a strictly positive smooth real function to be defined (see below).

The actual construction of a complete IKT of this type relies on classical statistical mechanics [17, 20], and its formulation comprises the usual axioms of CSM. It follows that, again, $f(\mathbf{x}, t; Z)$ must fulfill necessarily the Liouville equation (11) (which can be intended as a *phase-space Eulerian equation*). Here by construction $\mathbf{X}(\mathbf{x}, t; Z)$ is so defined that the fluid fields $\{\mathbf{V}(\mathbf{r}, t), p(\mathbf{r}, t)\}$ must be uniquely determined by the moment equations (16) and (17). It is important to stress that, by suitably specifying the functional class $\{f(\mathbf{x}, t; Z)\}$, the actual form of the vector field $\mathbf{X}(\mathbf{x}, t; Z)$ can be *explicitly constructed* [3] and *proven to be unique* [17, 20, 21]. Thus, a particular solution of Eq.(11) can be obtained by requiring that $f(\mathbf{x}, t; Z)$ for any $(\mathbf{x}, t) \in \Gamma \times I$ satisfies the two entropic principles:

- *Axiom of maximum entropy* (also known as principle of entropy maximization (PEM); Jaynes, 1957 [22]). This requires that: 1) $f(\mathbf{x}, t; Z)$ be a strictly-positive ordinary function which admits for all $t \in I$ the moment

$$S(f(t)) = - \int_{\Gamma} d\mathbf{x} f(\mathbf{x}, t; Z) \ln f(\mathbf{x}, t; Z), \quad (19)$$

$S(f(t))$ denoting the Boltzmann-Shannon (BS) entropy associated to the PDF $f(\mathbf{x}, t; Z)$; 2) that for all $t \in I$ the probability density $f(t) \equiv f(\mathbf{x}, t; Z)$ satisfies the *constrained maximal variational principle*

$$\delta S(f(t)) = 0, \quad (20)$$

with $f(t)$ required to satisfy the constraints (15) and (16) and (17);

- *Axiom of conservation of entropy* (also known as constant H-theorem [17]). This requires that there results identically for all $t \in I$

$$\frac{\partial}{\partial t} S(f(t)) = 0. \quad (21)$$

Imposing PEM requires necessarily that $f(\mathbf{x}, t; Z)$ must coincide with a local Maxwellian distribution [3, 17]

$$f_M(\mathbf{u}; p_1(\mathbf{r}, t)) = \frac{1}{\pi^2 v_{th}(\mathbf{r}, t)} \exp \left\{ -\frac{u^2}{v_{th}(\mathbf{r}, t)} \right\}, \quad (22)$$

where $\mathbf{u} = \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$,

$$v_{th}(\mathbf{r}, t) = \sqrt{2p_1(\mathbf{r}, t)/\rho_o}, \quad (23)$$

is the *thermal velocity due to the kinetic pressure*

$$p_1(\mathbf{r}, t) = p_0(t) + p(\mathbf{r}, t) - \phi(\mathbf{r}, t), \quad (24)$$

and the pseudo-pressure $p_0(t_o) > 0$ is uniquely determined in such a way to assure that in the time interval I the constant H-theorem [Eq.(21)] is satisfied identically. As a further consequence, one can prove [17] that the requirement that

$$f(\mathbf{x}, t; Z) \equiv f_M(\mathbf{u}; p_1(\mathbf{r}, t)) \quad (25)$$

be a solution of Eq.(11) implies necessarily that the vector field \mathbf{F} depends functionally on f_M and $\{Z\}$, hence it can be intended as a mean-field force [18]. As pointed out previously [17], the form of \mathbf{F} is actually non-unique. More precisely it can be cast in the form:

$$\mathbf{F}(\mathbf{x}, t; f_M; Z; a) = \mathbf{F}_0(\mathbf{x}, t; f_M; Z; a) + \mathbf{F}_1(\mathbf{x}, t; f_M; Z), \quad (26)$$

$$\mathbf{F}_0(\mathbf{x}, t; f_M; a) = \frac{1}{\rho_o} \mathbf{f}_R + \mathbf{D}(\mathbf{r}, t; a) + \nu \nabla^2 \mathbf{V}, \quad (27)$$

$$\mathbf{F}_1(\mathbf{x}, t; f_M) = \frac{1}{2} \mathbf{u} A(\mathbf{r}, t; f) + \Delta \mathbf{F}_1(\mathbf{x}, t; f_M), \quad (28)$$

$$\Delta \mathbf{F}_1(\mathbf{x}, t; f_M) = \frac{v_{th}}{2p_1} \nabla p_1 \left\{ \frac{u^2}{v_{th}} - \frac{3}{2} \right\}, \quad (29)$$

where $\mathbf{D}(\mathbf{r}, t; a)$ is the convective term

$$\mathbf{D}(\mathbf{r}, t; a) = \mathbf{u} \cdot \nabla \mathbf{V} b + a \nabla \mathbf{V} \cdot \mathbf{u}, \quad (30)$$

$b = 1 - a$, and $a \equiv a(\mathbf{r}, t)$ is an arbitrary scalar real function. Finally, the vector field $A(\mathbf{r}, t; f_M)$ is now defined as

$$\begin{aligned} A(\mathbf{r}, t; f_M) &\equiv \frac{1}{p_1} \frac{\partial}{\partial t} p_1 - \\ &- \frac{\rho_o}{p_1} \left[\frac{\partial}{\partial t} V^2/2 + \nabla \cdot (\mathbf{V} V^2/2) - \frac{1}{\rho_o} \mathbf{V} \cdot \mathbf{f} - \nu \mathbf{V} \cdot \nabla^2 \mathbf{V} \right] \equiv \\ &\equiv \frac{1}{p_1} \left[\frac{D}{Dt} p_1 + \frac{\rho_o V^2}{2} \nabla \cdot \mathbf{V} \right], \end{aligned} \quad (31)$$

where a term proportional to $\nabla \cdot \mathbf{V}$ [and hence identically vanishing when the isochoricity condition is satisfied; see Eq.(73) in Appendix A] has been added.

Here we stress that:

1. as previously pointed out [21] the convective term $\mathbf{D}(\mathbf{r}, t; a)$ is non-unique due to the indeterminacy of the parameter a . This implies also the non-uniqueness of the NS dynamical system (see next Section). The choice (9) for the parameter a then corresponds to the symmetry condition [21]

$$a = b = 1/2; \quad (32)$$

2. the complete IKT holds in principle for arbitrary choices of a , to be considered either a real constant or a smooth real function $a(\mathbf{r}, t)$;
3. as indicated elsewhere [3] the present theory can also be generalized in a straightforward way also to non-Maxwellian initial distribution functions $f(\mathbf{x}, t; Z)$.

3 - PHASE-SPACE LAGRANGIAN FORMULATION

The results of the previous Section permit us to formulate in a straightforward way the phase-space Lagrangian formulation for an incompressible NS fluid. The complete IKT implies, in fact, that there exists a phase-space classical dynamical system (called *NS dynamical system*)

$$\mathbf{x}_o \rightarrow \mathbf{x}(t) \equiv \{\mathbf{r}(t), \mathbf{v}(t)\} = T_{t,t_o} \mathbf{x}_o \equiv \chi(\mathbf{x}_o, t_o, t), \quad (33)$$

T_{t,t_o} denoting the corresponding evolution operator, which generates the time-evolution of the complete set of fluid fields, i.e., is such that

$$\{Z((\mathbf{r}(t), t))\} = T_{t,t_o} \{Z_o(\mathbf{r}_o)\} \quad (34)$$

[here $Z_o(\mathbf{r}_o) \equiv Z(\mathbf{r}_o, t_o)$ denotes the initial fluid fields evaluates at $t = t_o$; see Eq.(75)]. This means that the NS dynamical system is identified with the flow generated by the vector field \mathbf{X} defined by Eq.(14). In view of Eqs.(26)-(31) this must be considered of the form

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t; f; Z; a). \quad (35)$$

Hence, $\mathbf{x}(t)$ is obtained as the solution of the initial-value problem defined by Eq.(1), where, in particular, the mean-field force \mathbf{F} is expected to differ generally from \mathbf{F}_H as defined by Eq.(79) [see Appendix]. Moreover the state-vector $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ spans by assumption the phase space $\Gamma \equiv \Omega \times U$, while Ω and U denote appropriate configuration and velocity spaces. *The integral curves $\mathbf{x}(t)$ of Eq.(1) can therefore be interpreted as Lagrangian phase-space trajectories of tracer particles, whose dynamics is determined uniquely by the time evolution of the fluid.* The latter can also be cast in an equivalent *phase-space Lagrangian form*. By assumption the initial-value problem (1) is well-posed, namely it defines a suitably smooth diffeomorphism. Here, by construction:

- denoting by $\mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t)$ the solution of the initial-value problem (1), $\mathbf{x}_o = \chi(\mathbf{x}(t), t, t_o)$ is its inverse. Both are assumed to be suitably smooth functions of the relevant parameters;
- both $\mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t)$ and $\mathbf{x}_o = \chi(\mathbf{x}(t), t, t_o)$ identify admissible LP's of the dynamical system;
- $\mathbf{r}(t)$ is the Lagrangian trajectory which belongs to the fluid domain Ω ;

- $\mathbf{v}(t)$ and $\mathbf{F}(\mathbf{r}(t), t; f)$ are respectively the Lagrangian velocity and acceleration, both spanning the vector space \mathbb{R}^3 . In particular, $\mathbf{F}(\mathbf{r}(t), t; f)$, which is defined by Eqs.(26)-(31), and depends functionally on the kinetic probability density $f(\mathbf{x}, t)$, is the *Lagrangian acceleration which corresponds to an arbitrary kinetic probability density $f(\mathbf{x}, t)$* ;
- $f(\mathbf{x}, t)$ is a particular solution of the inverse kinetic equation (11).

It follows that the Jacobian $[J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right|]$ of the map $\mathbf{x}_o \rightarrow \mathbf{x}(t)$, which is generated by Eq.(1) for $f \equiv f_M(\mathbf{x}, t)$ reads

$$J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| = \frac{v_{th}(t)}{v_{th}(t_o)} \exp \left\{ -\frac{u^2(t_o)}{v_{th}(t_o)} + \frac{u^2(t)}{v_{th}(t)} \right\}. \quad (36)$$

Furthermore, it is immediate to prove that the inverse kinetic equation can be written in Lagrangian form. This yields the so-called *integral Liouville equation*

$$J(\mathbf{x}(t), t) f(\mathbf{x}(t), t; Z) = f(\mathbf{x}_o, t_o, Z), \quad (37)$$

with $f(\mathbf{x}_o, t_o, Z)$ denoting the initial PDF and $J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right|$ the Jacobian of the map (33). As a consequence, for all $(\mathbf{x}, t) \in \bar{\Gamma} \times I$ (with $\bar{\Gamma} \equiv \bar{\Omega} \times U$) the PDF $f(\mathbf{x}, t; Z)$ can be represented explicitly in terms of the initial PDF and reads

$$f(\mathbf{x}, t; Z) = \frac{1}{J(t; Z)} f(\chi(\mathbf{x}, t, t), t_o, Z), \quad (38)$$

where $J(t; Z) \equiv J(\mathbf{x}, t)$, $\chi(\mathbf{x}, t, t) \equiv T_{t,t_o}^{-1} \mathbf{x}$ and $\mathbf{x} \equiv \mathbf{x}(t)$. In particular, in terms of Eq.(38) the fluid fields can be explicitly evaluated, yielding:

$$\mathbf{V}(\mathbf{r}, t) = \int_U d\mathbf{v} \mathbf{v} \frac{1}{J(t; Z)} f(\chi(\mathbf{x}, t, t), t_o, Z), \quad (39)$$

$$p(\mathbf{r}, t) = \int_U d\mathbf{v} \frac{1}{J(t; Z)} f(\chi(\mathbf{x}, t, t), t_o, Z) \quad (40)$$

$$\left[\frac{\rho_o u^2}{3} - p_o(t) - \phi(\mathbf{r}, t) \right].$$

From the *mathematical standpoint* main consequences of the IKT representation are that: 1) the Lagrangian formulation (of IKT) is uniquely specified by the proper definition of a suitable family of phase-space LP's; 2) Eq.(38) uniquely specifies the time-evolution of the Eulerian PDF, $f(\mathbf{x}(t), t)$, which is represented in terms of the initial distribution function $f_o(\mathbf{x}_o)$ and the LP's defined

by the INSE dynamical system; 3) the time-evolution of the fluid fields $\{\rho = \rho_o > 0, \mathbf{V}, p \geq 0, T > 0, S_T\}$ is uniquely specified via the PDF $f(\mathbf{x}(t), t)$; 4) Eq.(38) also provides the connection between Lagrangian and Eulerian viewpoints. In fact the Eulerian PDF, $f(\mathbf{x}, t)$, is simply obtained from Eq.(38) by letting $\mathbf{x} = \mathbf{x}(t)$ in the same equation. As a result, the Eulerian and Lagrangian formulations of IKT, and hence of the underlying moment (i.e., fluid) equations, are manifestly equivalent. From the *physical viewpoint*, it is worth mentioning that the LP's here defined can be interpreted as phase-space trajectories of the particles of the fluid, to be considered as a set of "classical molecules", i.e., point particles with prescribed mass, which interact with the fluid only via the action of a suitable mean-field force kind. The ensemble motion of these particles has been defined in such a way that it uniquely determines the time evolution *both* of the kinetic distribution functions *and* of the relevant fluid fields which characterize the NS fluid.

4 - THE DYNAMICS OF THERMAL TRACER PARTICLES

An interesting issue is the search of *possible exact particular solutions* of the Lagrangian equations (1). The general solution of Eq.(1) implies

$$\mathbf{v}(t) = \mathbf{V}(\mathbf{r}(t), t) + \mathbf{n}(\mathbf{r}(t), t)u(\mathbf{r}(t), t), \quad (41)$$

where $\mathbf{n}(\mathbf{r}(t), t)$ is the rotating unit vector which satisfies the initial-value problem Eq.(6). For arbitrary tracer-particle solutions it is obvious that $\mathbf{n}(\mathbf{r}(t), t)$ is an arbitrary unit vector belonging to the unit sphere $[S(U)]$ of velocity space $U \equiv \mathbb{R}^3$. Hence, by definition, $S(U)$ is a set of dimension 2. In fact, Eq.(1) yields for the relative velocity $\mathbf{u} = \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$ the equation

$$\frac{d}{dt}\mathbf{u} = \mathbf{F}_u(\mathbf{x}, t; f_M; Z; a), \quad (42)$$

$$\begin{aligned} \mathbf{F}_u(\mathbf{x}, t; f_M; Z; a) = & a \nabla \mathbf{V} \cdot \mathbf{u} - a \mathbf{u} \cdot \nabla \mathbf{V} + \\ & + \frac{\mathbf{u}}{2p_1} \left[\frac{D}{Dt} p_1 + \frac{\rho_o V^2}{2p_1} \nabla \cdot \mathbf{V} \right] + \\ & + \frac{v_{th}}{2p_1} \nabla p_1 \left\{ \frac{u^2}{v_{th}} - \frac{1}{2} \right\}. \end{aligned} \quad (43)$$

It follows

$$\frac{d}{dt} \frac{u^2}{2} = \frac{u^2}{2} \frac{1}{p_1} \left[\frac{d}{dt} p_1 + \frac{\rho_o V^2}{2} \nabla \cdot \mathbf{V} \right] - \frac{1}{2\rho_o} \mathbf{u} \cdot \nabla p_1. \quad (44)$$

Furthermore, Eq.(42), in view of Eq.(44), implies for \mathbf{n}

$$\begin{aligned} \frac{d}{dt} \mathbf{n}(\mathbf{r}, t) = & \mathbf{n}(\mathbf{r}, t) \times \boldsymbol{\Omega}(\mathbf{r}, t) - \\ & - \frac{1}{\rho_o} \left\{ \frac{u^2}{v_{th}} - \frac{1}{2} \right\} \mathbf{n}(\mathbf{r}, t) \times [\mathbf{n}(\mathbf{r}, t) \times \nabla p_1]. \end{aligned} \quad (45)$$

There it follows:

Theorem 1 - Existence of thermal tracer particles

If the fluid fields $\{\mathbf{V}(\mathbf{r}, t), p(\mathbf{r}, t)\}$ are classical solutions of INSE [see Eqs.(72)-(76) in Appendix A], then a particular solution $\mathbf{x}(t) \equiv \mathbf{x}_{Th}(t)$ of Eq.(1) is provided by Eqs.(2)-(7), subject to the orthogonality and kinematic constraints (8) and (10).

PROOF

First we notice, invoking Eq.(44) and assuming that $\mathbf{V}(\mathbf{r}, t)$ satisfies in particular the isochoricity condition [Eq.(74), in Appendix A], that Eq.(5) implies necessarily

$$\frac{d}{dt} \frac{u^2}{2} = \frac{u^2}{2} \frac{1}{p_1} \frac{d}{dt} p_1. \quad (46)$$

It follows that the unit vector $\mathbf{n}(\mathbf{r}, t)$ must satisfy identically Eq.(8). On the other hand, thanks to Eq.(6), according to Eq.(45), $\mathbf{n}(\mathbf{r}, t)$ is a rotating unit vector with angular velocity $\boldsymbol{\Omega}(\mathbf{r}, t)$ defined by Eq.(7). Hence, the condition of orthogonality Eq.(8) requires

$$\frac{d}{dt} \mathbf{n} \cdot \nabla p_1 + \mathbf{n} \cdot \frac{d}{dt} \nabla p_1 = 0 \quad (47)$$

too. Eq.(6) then implies manifestly also Eq.(10), which can be viewed as a kinematic constraint for the parameter a , to be considered as dynamical function. If ω is non-vanishing and the scalar product $\mathbf{n} \cdot \omega$ is defined so that it is $\neq 0$ it is obvious that the previous equation can always be satisfied. On the contrary, if locally $\omega = \mathbf{0}$, this can happen either if 1) $\xi = \mathbf{0}$, 2) $\nabla p_1 = 0$ or 3) $\xi \parallel \nabla p_1$.

In the first case, it follows Eq.(6) reduces to $\frac{d}{dt} \mathbf{n} = \mathbf{0}$ which requires also $\mathbf{n} \cdot \nabla p_1 = 0$ and moreover \mathbf{V} is potential, i.e., there exists a real scalar $G(\mathbf{r}, t)$ such that $\mathbf{V} = \nabla G(\mathbf{r}, t)$. This implies that $G(\mathbf{r}, t)$ must satisfy the PDE

$$\frac{\partial}{\partial t} \nabla G + \frac{1}{2} \nabla (|\nabla G|^2) + \nabla p_1 = 0, \quad (48)$$

namely locally it must be

$$\mathbf{n} \cdot \left[\frac{\partial}{\partial t} \nabla G + \frac{1}{2} \nabla (|\nabla G|^2) \right] = 0 \quad (49)$$

too. In the second case the direction of \mathbf{n} is arbitrary so that it can always be chosen in such a way to satisfy Eq.(6). Finally, in the third case if \mathbf{n} is orthogonal ∇p_1 it remains always so by construction thanks to Eq.(6). Let us now impose the constraints (8), (10) and consider an arbitrary initial condition of the form

$$\mathbf{x}(t_o) = \{\mathbf{r}(t_o) = \mathbf{r}_o, \mathbf{v}(t_o) = \mathbf{v}_o\}, \quad (50)$$

with

$$\mathbf{v}_o = \mathbf{V}(\mathbf{r}_o, t_o) + \mathbf{n}(\mathbf{r}_o, t_o)u(t_o), \quad (51)$$

$$u(t_o) = v_{th}(\mathbf{r}_o, t_o) \equiv \sqrt{\frac{2p_1(\mathbf{r}_o, t_o)}{\rho_o}}, \quad (52)$$

$$\mathbf{n}(\mathbf{r}_o, t_o) \cdot \nabla_o p_1(\mathbf{r}_o, t_o) = 0 \quad (53)$$

Here $\{\mathbf{V}(\mathbf{r}_o, t_o), p(\mathbf{r}_o, t_o)\}$ are arbitrary initial fluid fields consistent with INSE (see Appendix A), while $p_1(\mathbf{r}_o, t_o)$ and $\nabla_o p_1(\mathbf{r}_o, t_o)$ are defined according to Eqs.(24). Finally, $\mathbf{n}(\mathbf{r}_o, t_o) \equiv \mathbf{n}_o$ is an arbitrary unit vector orthogonal to $\nabla_o p_1(\mathbf{r}_o, t_o)$. It is obvious that the particular solution $\mathbf{x}(t)$ which satisfies the previous initial conditions (50)-(53) coincides necessarily with $\mathbf{x}_{TT}(t)$ defined by Eqs.(2)-(7). Q.E.D.

Let us analyze the physical interpretation of the theorem.

A basic consequence is that an arbitrary tracer particle, subject to the kinematic constraint (10) and fulfilling the initial conditions (50)-(53) is necessarily a thermal tracer particle for all $\in I$. Hence, at time t its relative velocity is

$$\mathbf{u}(\mathbf{r}(t), t) = \mathbf{n}(\mathbf{r}(t), t) \sqrt{\frac{2p_1(\mathbf{r}(t), t)}{\rho_o}}, \quad (54)$$

with \mathbf{n} denoting a rotating unit vector in accordance to Eqs.(6) and (7). The consequence is that tracer particles with initial velocity (50)-(53) behave as *thermal particles*, i.e., at any time t their relative velocity [with respect to the local fluid element] is solely determined by the kinetic pressure $p_1(\mathbf{r}, t)$ [in turn related via Eq.(24) to the fluid pressure]. The direction $[\mathbf{n}(\mathbf{r}, t)]$ of their relative velocity $\mathbf{u}(\mathbf{r}, t)$ depends, instead, on the angular velocity $\boldsymbol{\Omega}(\mathbf{r}, t)$ defined by Eq.(7) and depending only on the fluid vorticity. However, in order to assure the condition of orthogonality (8) for the relative velocity, the kinematic constraint (10) must be satisfied so that both $\boldsymbol{\Omega}(\mathbf{r}, t)$ and the mean-field force \mathbf{F} are both uniquely determined. This implies that \mathbf{u} necessarily belongs to the 1-dimensional subset of velocity space $U_{th} \subset U$ for which the orthogonality condition Eq.(8) holds.

5 - THERMAL TRACER PARTICLE 1-D STATISTICS

Apart the physical insight afforded by the qualitative behavior of tracer-particle dynamics, the existence of the particular solution defined by Eqs.(2)-(7) is important in order to establish a phase-space statistical description for incompressible NS fluids. In this section we intend to prove that the dynamics of thermal tracer particles is *actually sufficient to determine uniquely the time evolution of the fluid fields*. In other words, to characterize completely the statistical behavior of the fluid it is sufficient, in principle, to determine *only* the dynamics of these tracer particles.

In view of the general form of the solution for thermal tracer particles [see Eqs.(2)-(7)], it is obvious that the statistics depends only on the unit vector $\mathbf{n}(\mathbf{r}, t)$ which defines the direction of the particle relative velocity with respect to the fluid [see Eq.(6)]. Thanks to

the orthogonality constraint (8) placed on the dynamics of thermal tracer particles it follows that $\mathbf{n}(\mathbf{r}, t)$ is orthogonal to $\nabla p_1(\mathbf{r}, t) \equiv \hat{e}_z |\nabla p_1(\mathbf{r}, t)|$, i.e., $\mathbf{n}(\mathbf{r}, t)$ must belong to a family of coplanar unit vectors which are tangent to the isobaric surface $\{p_1(\mathbf{r}, t) = \text{const.}\}$. Hence, the only admissible velocity-space statistics is necessarily one-dimensional (since it is defined on the subset of velocity space U_{th} define above). This is obtained by considering $\mathbf{n}(\mathbf{r}, t)$ as a *stochastic* unit vector characterized by a suitable stochastic PDF g . It is obvious that g is necessarily related to the PDF introduced in the complete IKT described in the previous section, i.e., to $f(\mathbf{x}, t; Z)$. In case (25), by parametrizing \mathbf{n} in cylindrical coordinates

$$\mathbf{n} = \mathbf{n}(\mathbf{r}, \vartheta, t) = \hat{e}_x \cos \vartheta + \hat{e}_y \sin \vartheta. \quad (55)$$

it follows that g must be necessarily identified with the uniform distribution

$$g = 1/2\pi. \quad (56)$$

Thus, introducing the stochastic average

$$\langle \rangle = \int_0^{2\pi} d\vartheta g \quad (57)$$

[where the angle-integration is performed at constant $\mathbf{r} \equiv \mathbf{r}(t)$ and t] it follows by definition that

$$\langle 1 \rangle = 1, \quad (58)$$

$$\langle \mathbf{n} \rangle = \mathbf{0}, \quad (59)$$

$$\langle \mathbf{n}\mathbf{n} \rangle = \frac{1}{2} \mathbf{I}_2 \equiv \frac{1}{2} [\hat{e}_x \hat{e}_x + \hat{e}_y \hat{e}_y]. \quad (60)$$

As a consequence it is immediate to show that:

- the fluid fields $\{\mathbf{V}(\mathbf{r}, t), p(\mathbf{r}, t)\}$ can be related to suitable stochastic averages of thermal tracer particle dynamics;
- similarly, by taking the stochastic average of the Lagrangian equations (1) and (44), the stochastic-averaged equations

$$\frac{d}{dt} \langle \mathbf{v}(t) \rangle = \langle \mathbf{F}(\mathbf{x}, t; f_M, Z) \rangle, \quad (61)$$

$$\frac{1}{2} \frac{d}{dt} \langle u^2(t) \rangle = \langle \mathbf{u}(t) \cdot \mathbf{F}_u(\mathbf{x}, t; f_M, Z) \rangle \quad (62)$$

can be evaluated explicitly for thermal tracer particles.

The following result holds.

Theorem 2 - 1-D statistics of thermal tracer particles

In validity of THM.1, let us assume that $\mathbf{x}(t) = \{\mathbf{r}(t), \mathbf{v}(t)\}$ is an arbitrary phase-space trajectory of a thermal tracer particle [i.e., a solution of the form given

by Eqs.(2)-(7)]. Then, due to (56) and (57), it follows that:

1) the stochastic averages $\langle \mathbf{v}(t) \rangle$ and $\langle u(t)^2 \rangle$ are related to the local values of the NS fluid fields $\{\mathbf{V}(\mathbf{r}, t), p(\mathbf{r}, t)\}$, namely it results

$$\langle \mathbf{v}(t) \rangle = \mathbf{V}(\mathbf{r}, t), \quad (63)$$

$$\frac{\rho_o}{2} \langle u(t)^2 \rangle - p_0(t) + \phi(\mathbf{r}, t) = p(\mathbf{r}, t), \quad (64)$$

where $p_0(t)$ and $\phi(\mathbf{r}, t)$ are defined by (24),(80) [see Appendix A] and the constant H-theorem (21);

2) equations (61) and (62) deliver respectively the NS equation and the isochoricity condition [see Eqs.(74) and Eq.(73) in Appendix A].

PROOF

The proof of Eqs.(63) and (64) follows immediately from Eqs.(58)-(60) and Eqs.(4)-(6).

To evaluate the stochastic average of Eq.(61) we first recall the identity

$$\frac{d}{dt} \mathbf{v} = \frac{\partial}{\partial t} \mathbf{V} + (\mathbf{V} + \mathbf{n}u) \cdot \nabla \mathbf{V} + \frac{d}{dt} \mathbf{u}. \quad (65)$$

Invoking Eqs.(42) and (43) it follows

$$\begin{aligned} \left\langle \frac{d}{dt} \mathbf{v} \right\rangle &= \frac{\partial}{\partial t} \mathbf{V}(\mathbf{r}, t) + \mathbf{V} \cdot \nabla \mathbf{V} + \\ &+ \left\langle \frac{d}{dt} \mathbf{u} \right\rangle = \mathbf{F}_H + \langle \mathbf{F}_u(\mathbf{x}, t; f_M; Z) \rangle, \end{aligned} \quad (66)$$

where \mathbf{F}_H is given by Eq.(79) [see Appendix A] and

$$\langle \mathbf{F}_u(\mathbf{x}, t; f_M; Z) \rangle = \frac{v_{th}}{2p_1} \nabla p_1 \left\{ \frac{u^2}{v_{th}} - \frac{1}{2} \right\} = 0. \quad (67)$$

Hence Eq.(66) [and Eq.(61)] coincides manifestly with the NS equation (74).

In a similar way it is immediate to evaluate Eq.(62). From Eqs.(4) and (5), in validity of the constraint (8), one obtains in fact:

$$\left\langle \frac{d}{dt} \frac{u^2}{2} \right\rangle = \langle \mathbf{u} \cdot \mathbf{F}_u(\mathbf{x}, t; f_M; Z) \rangle, \quad (68)$$

while under the same conditions from Eq. (43) it follows

$$\begin{aligned} \left\langle \mathbf{u} \cdot \frac{d}{dt} \mathbf{v} \right\rangle &= \left\langle \mathbf{u} \cdot \frac{\partial}{\partial t} \mathbf{V} \right\rangle + \langle (\mathbf{V} + \mathbf{u}) \cdot \nabla \mathbf{V} \cdot \mathbf{u} \rangle + \left\langle \frac{d}{dt} \frac{u^2}{2} \right\rangle = \\ &= \frac{1}{2} \frac{2p_1}{\rho_o} \nabla \cdot \mathbf{V} + \frac{D}{Dt} \frac{p_1}{\rho_o} \end{aligned}$$

$$\left\langle \frac{d}{dt} \frac{u^2}{2} \right\rangle = \frac{D}{Dt} \frac{p_1(\mathbf{r}, t)}{\rho_o}$$

$$\langle \mathbf{u} \cdot \mathbf{F}_u(\mathbf{x}, t; f_M; Z) \rangle = \frac{1}{\rho_o} \left[\frac{D}{Dt} p_1 + \frac{\rho_o V^2}{2} \nabla \cdot \mathbf{V} \right]. \quad (69)$$

Eqs.(62) and (68) necessarily yield the identity

$$\frac{V^2}{2} \nabla \cdot \mathbf{V} = 0. \quad (70)$$

Hence, provided $V^2 \neq 0$, also the isochoricity condition [see Eq.(73) in Appendix A] is identically fulfilled. Q.E.D.

6 - CONCLUDING REMARKS

In this paper properties of the NS dynamical system, advancing in time the state of an incompressible NS fluid, have been investigated. This refers, in particular, to the proof of the existence of a particular subset of phase-space trajectories (solutions of the NS dynamical) representing the dynamics of so-called *thermal tracer particles*. The result has been reached by imposing suitable constraints to the direction of \mathbf{u} (relative velocity) of tracer particles and on the parameter a which characterizes its time evolution. In turn, the same constraint determines uniquely also the mean-field force \mathbf{F} which defines the NS dynamical system. As a main consequence, we have proven (THM.1) that there exists a family of particular solutions $\mathbf{x}(t) = \{\mathbf{r}(t), \mathbf{v}(t)\}$ which have the following features:

1. the magnitude of the relative velocity with respect to the fluid, $\mathbf{u} = \mathbf{n}u$, is determined only by the kinetic pressure p_1 (i.e., tracer particles of this type behave as thermal particles);
2. the direction of the relative velocity \mathbf{n} is a rotating unit vector. Its angular velocity is due to the fluid vorticity [see Eq.(7)];
3. the dimension of the subset of the velocity space $U_{th} \subseteq U$ spanned by these solutions is $\dim(U_{th}) = 1$; hence their phase space $\Gamma_{th} = \Omega \times U_{th}$ is of dimension $\dim(\Gamma_{th}) = 4$.

Furthermore, the 1-D velocity-space statistics of thermal tracer particles has been investigated (Section 5). As a main consequence (see THM.2) :

1. The state of thermal tracer particles has been proven to determine uniquely, via a suitable statistics [namely by means of the stochastic averages (63) and (64)], the complete set of fluid fields $\{\mathbf{V}, p\}$. As a consequence the state of the fluid has been proven to be uniquely related to that of the thermal tracer particles.
2. The complete set of fluid equations has been represented in terms of stochastic averages of the phase-space Lagrangian equations for thermal tracer particles (61) and (62).

These conclusions show that a *phase-space Lagrangian dynamics for an incompressible NS fluid based exclusively on the dynamics of thermal tracer particles is possible*. The result appears relevant because of its outstanding physical interpretation. The phase-space Lagrangian dynamics here determined permits, in fact, to advance in time self-consistently the fluid fields, i.e., in such a way that they satisfy identically the required set of fluid equations. The proof of this statement is straightforward and is given in Appendix B.

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APPENDIX A - DETERMINISTIC DESCRIPTION OF NS FLUIDS

In fluid dynamics the state of a fluid is assumed to be prescribed by an appropriate set of suitably smooth functions $\{Z\} \equiv \{Z_i, i = 1, n\}$ denoted as *fluid fields*. These are required to be real functions of the form

$$Z_i = Z_i(\mathbf{r}, t), \quad (71)$$

with \mathbf{r} and t spanning respectively the sets Ω (configuration space) and I (time axis). In particular the functions $Z_i(\mathbf{r}, t)$ are assumed at least continuous in all points of a closed set $\bar{\Omega} \times \bar{I}$ (extended configuration space), with $\bar{\Omega}$ closure of Ω . In the remainder we shall require that:

1. Ω is a bounded subset of the Euclidean space E^3 on \mathbb{R}^3 ;
2. I is identified, when appropriate, either with a bounded time interval, i.e., $I =]t_0, t_1[\subseteq \mathbb{R}$, or with the real axis \mathbb{R} ;
3. in the open set $\Omega \times I$ the functions $\{Z\}$ are assumed to be solutions of an appropriate closed set of PDE's, denoted as *fluid equations*;
4. by assumption, these equations together with appropriate initial and boundary conditions are required to define a well-posed problem with unique strong solution defined everywhere in $\Omega \times I$.

The deterministic description of an incompressible NS fluid is provided by the fluid fields $\{Z\} \equiv \{\mathbf{V}, p\}$, with

$\mathbf{V}(\mathbf{r}, t)$ and $p(\mathbf{r}, t) \geq 0$ denoting respectively the deterministic fluid velocity and pressure and by the *incompressible NS equations* (INSE):

$$\rho = \rho_o, \quad (72)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (73)$$

$$N\mathbf{V} = 0, \quad (74)$$

$$Z(\mathbf{r}, t_o) = Z_o(\mathbf{r}), \quad (75)$$

$$Z(\mathbf{r}, t)|_{\partial\Omega} = Z_w(\mathbf{r}, t)|_{\partial\Omega}. \quad (76)$$

Here N is the NS nonlinear operator

$$N\mathbf{V} = \frac{D}{Dt}\mathbf{V} - \mathbf{F}_H, \quad (77)$$

$$\frac{D}{Dt}\mathbf{V} \equiv \frac{\partial}{\partial t}\mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}, \quad (78)$$

$$\mathbf{F}_H \equiv -\frac{1}{\rho_o}\nabla p + \frac{1}{\rho_o}\mathbf{f} + \nu \nabla^2 \mathbf{V}, \quad (79)$$

with $\frac{D}{Dt}\mathbf{V}$, \mathbf{F}_H and $\mathbf{f}(\mathbf{r}, t)$ denoting respectively the Lagrangian fluid acceleration, the total force per unit mass and the volume force density acting on the fluid, while $\rho_o > 0, \nu > 0$ the constant mass density and the constant kinematic viscosity. In particular, \mathbf{f} is assumed of the form

$$\mathbf{f} = -\nabla \phi(\mathbf{r}, t) + \mathbf{f}_R, \quad (80)$$

$\phi(\mathbf{r}, t)$ denoting a suitable scalar potential, so that the first two force terms [in Eq.(79)] can be represented as $-\nabla p + \mathbf{f} = -\nabla p_r + \mathbf{f}_R$, with

$$p_r(\mathbf{r}, t) = p(\mathbf{r}, t) - \phi(\mathbf{r}, t), \quad (81)$$

denoting the *reduced fluid pressure*. Moreover, Eqs. (72), (73), (74), (75) and (76) are respectively the incompressibility, isochoricity, NS equations and the initial and Dirichlet boundary conditions for $\{Z\}$, with $\{Z_o(\mathbf{r})\}$ and $\{Z_w(\mathbf{r}, t)|_{\partial\Omega}\}$ suitably prescribed deterministic initial and boundary-value fluid fields, defined respectively at the initial time $t = t_o$ and on the boundary $\partial\Omega$.

APPENDIX B - TIME EVOLUTION OF THE NS FLUID FIELDS

Let us now show that the phase-space dynamics of thermal tracer particles determines uniquely the complete set of fluid fields $\{\mathbf{V}, p\}$. To prove the statement, let us assume that (at the initial time $t_o \in I$) the initial values of the fluid fields $\{\mathbf{V}(\mathbf{r}_o, t_o), p(\mathbf{r}_o, t_o)\}$ are prescribed for all $\mathbf{r}_o \in \bar{\Omega}$. Let us require that at any time $t > t_o$ (with $t \in I$) an arbitrary position \mathbf{r} (this includes also the case in which \mathbf{r} belongs to the boundary $\partial\Omega$) is reached (at least) by two thermal tracer particles with phase-space trajectories. Denoting by $\mathbf{x}_i(t) = \{\mathbf{r}_i(t) = \mathbf{r}, \mathbf{v}_i(t)\}$

(for $i = 1, 2$) their phase-space states, Eqs.(2)-(5) then imply that for $i = 1, 2$ it must result

$$\mathbf{v}_i(t) = \mathbf{V}(\mathbf{r}, t) + \mathbf{n}_i(\mathbf{r}, t)v_{th}(\mathbf{r}, t), \quad (82)$$

where the unit vectors $\mathbf{n}_i(\mathbf{r}, t)$ are, by definition, solutions of the initial value problem (6) for the initial conditions $\mathbf{n}(\mathbf{r}_i(t_o), t_o) = \mathbf{n}_i(\mathbf{r}_{oi}, t_o)$ and $v_{th}(\mathbf{r}, t) = \sqrt{\frac{2p_1(\mathbf{r}, t)}{\rho_o}}$. If we require, in particular, that $\mathbf{n}_1(\mathbf{r}, t) = -\mathbf{n}_2(\mathbf{r}, t)$ the same conclusion is immediate, because then

$$\mathbf{V}(\mathbf{r}, t) = \frac{\mathbf{v}_1(t) + \mathbf{v}_2(t)}{2}, \quad (83)$$

$$v_{th}(\mathbf{r}, t) = \frac{|\mathbf{v}_1(t) - \mathbf{v}_2(t)|}{2}, \quad (84)$$

where the second equation yields $p(\mathbf{r}, t)$ in terms of Eq.(24). Let us now assume, instead, more generally that there results $\mathbf{n}_1(\mathbf{r}, t) \neq -\mathbf{n}_2(\mathbf{r}, t)$. In this case it follows

$$v_{th}(\mathbf{r}, t) = \frac{|\mathbf{v}_1(t) - \mathbf{v}_2(t)|}{|\mathbf{n}_1(\mathbf{r}, t) - \mathbf{n}_2(\mathbf{r}, t)|}, \quad (85)$$

$$\mathbf{V}(\mathbf{r}, t) = \frac{\mathbf{v}_1(t) + \mathbf{v}_2(t)}{2} - [(\mathbf{n}_1(\mathbf{r}, t) + \mathbf{n}_2(\mathbf{r}, t))v_{th}(\mathbf{r}, t)]. \quad (86)$$

Therefore, again, the fluid fields are uniquely determined at any $(\mathbf{r}, t) \in \overline{\Omega} \times I$.

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